ASYMPTOTICS OF TOEPLITZ MATRICES WITH SYMBOLS IN SOME GENERALIZED KREIN ALGEBRAS

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To the memory of Mark Krein (1907–1989)

ABSTRACT. Let $\alpha, \beta \in (0, 1)$ and

$$K^{\alpha,\beta} := \left\{ a \in L^\infty(\mathbb{T}) : \sum_{k=1}^\infty |\widehat{a}(-k)|^2 k^{2\alpha} < \infty, \ \sum_{k=1}^\infty |\widehat{a}(k)|^2 k^{2\beta} < \infty \right\}.$$

Mark Krein proved in 1966 that $K^{1/2,1/2}$ forms a Banach algebra. He also observed that this algebra is important in the asymptotic theory of finite Toeplitz matrices. Ten years later, Harold Widom extended earlier results of Gabor Szegő for scalar symbols and established the asymptotic trace formula

$$\text{trace } f(T_n(a)) = (n+1)G_f(a) + E_f(a) + o(1) \quad \text{as } n \to \infty$$

for finite Toeplitz matrices $T_n(a)$ with matrix symbols $a \in K^{1/2,1/2}_{N \times N}$. We show that if $\alpha + \beta \geq 1$ and $a \in K^{\alpha,\beta}_{N \times N}$, then the Szegő-Widom asymptotic trace formula holds with $o(1)$ replaced by $o(n^{1-\alpha-\beta})$.

1. INTRODUCTION AND THE MAIN RESULT

For $1 \leq p \leq \infty$, let $L^p := L^p(\mathbb{T})$ and $H^p := H^p(\mathbb{T})$ be the standard Lebesgue and Hardy spaces on the unit circle $\mathbb{T}$, respectively. Denote by $\{\widehat{a}(k)\}_{k \in \mathbb{Z}}$ the sequence of the Fourier coefficients of a function $a \in L^1(\mathbb{T})$,

$$\widehat{a}(k) = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-ik\theta} d\theta \quad (k \in \mathbb{Z}).$$

For $\alpha, \beta \in (0, 1)$, put

$$K^{\alpha,0} := \left\{ a \in L^\infty(\mathbb{T}) : \sum_{k=1}^\infty |\widehat{a}(-k)|^2 k^{2\alpha} < \infty, \right\},$$

$$K^{0,\beta} := \left\{ a \in L^\infty(\mathbb{T}) : \sum_{k=1}^\infty |\widehat{a}(k)|^2 k^{2\beta} < \infty \right\},$$

$$K^{\alpha,\beta} := K^{\alpha,0} \cap K^{0,\beta}.$$

It was Mark Krein [18] who first discovered that $K^{1/2,1/2}$ forms a Banach algebra under pointwise multiplication and the norm

$$\|a\|_{1/2,1/2} := \|a\|_{L^\infty} + \left(\sum_{k=-\infty}^{\infty} |\widehat{a}(k)|^2 (|k|+1) \right)^{1/2}.$$

By the same method, one can show that if $\alpha, \beta \in [1/2, 1)$, then $K^{\alpha,0}$ and $K^{0,\beta}$ are Banach algebras under pointwise multiplication and the norms

$$\|a\|_{\alpha,0} := \|a\|_{L^\infty} + \left(\sum_{k=0}^{\infty} |\widehat{a}(-k)|^2 (k+1)^{2\alpha} \right)^{1/2},$$

$$\|a\|_{0,\beta} := \|a\|_{L^\infty} + \left(\sum_{k=0}^{\infty} |\widehat{a}(k)|^2 (k+1)^{2\beta} \right)^{1/2},$$

respectively. Further, if $\max\{\alpha, \beta\} \geq 1/2$, then $K^{\alpha,\beta}$ is a Banach algebra under pointwise multiplication and the norm

$$\|a\|_{\alpha,\beta} := \|a\|_{L^\infty} + \left(\sum_{k=0}^{\infty} |\widehat{a}(-k)|^2 (k+1)^{2\alpha} \right)^{1/2} + \left(\sum_{k=0}^{\infty} |\widehat{a}(k)|^2 (k+1)^{2\beta} \right)^{1/2}$$

(see [4, Chap. 4] and also [6, Sections 10.9–10.11] and [2, Theorem 1.3]). In these sources even more general algebras are considered. The algebra $K^{1/2,1/2}$ is referred to as the *Krein algebra*. The algebras $K^{\alpha,0}$, $K^{0,\beta}$, and $K^{\alpha,\beta}$ will be called *generalized Krein algebras*.

Suppose $N \in \mathbb{N}$. For a Banach space X, let X_N and $X_{N\times N}$ be the spaces of vectors and matrices with entries in X, respectively. Let I be the identity operator, P be the Riesz projection of L^2 onto H^2, $Q := I - P$, and define I, P, and Q on L_N^2 elementwise. For $a \in L_{N\times N}^\infty$ and $t \in \mathbb{T}$, put $\widetilde{a}(t) := a(1/t)$ and $(Ja)(t) := t^{-1}\widetilde{a}(t)$. Define *Toeplitz operators*

$$T(a) := PaP|\operatorname{Im} P, \quad T(\widetilde{a}) := JQaQJ|\operatorname{Im} P$$

and *Hankel operators*

$$H(a) := PaQJ|\operatorname{Im} P, \quad H(\widetilde{a}) := JQaP|\operatorname{Im} P.$$

The function a is called the *symbol* of $T(a)$, $T(\widetilde{a})$, $H(a)$, $H(\widetilde{a})$. We are interested in the asymptotic behavior of *finite block Toeplitz matrices*

$$T_n(a) := \left(\widehat{a}(j-k) \right)_{j,k=0}^{n}$$

generated by (the Fourier coefficients of) the symbol a as $n \to \infty$. It should be noted that asymptotics of Toeplitz matrices was one of the topics of Mark Krein's interests. In particular, he proved [18] that $K^{1/2,1/2}$ is an optimal smoothness class for the validity of the strong Szegő limit theorem for scalar positive symbols (independently this result was obtained by Devinatz [8]; for an extension of this result to matrix positive definite symbols, see Böttcher and Silbermann [3]). Many results about asymptotic properties of $T_n(a)$ as $n \to \infty$ are contained in the books by Grenander and Szegő [11], Böttcher and Silbermann [4, 5, 6], Hagen, Roch, and Silbermann [12], Simon [24], and Böttcher and Grudsky [1].

Let $\operatorname{sp} A$ denote the spectrum of an operator A. If f is an analytic function in an open neighborhood of $\operatorname{sp} A$, then we will simply say that f is analytic on $\operatorname{sp} A$.

We assume that the reader is familiar with basics of trace class operators and their operator determinants (see Gohberg and Krein [10, Chap. 3 and 4] or Section 3). If A is a trace class operator, then $\operatorname{trace} A$ denotes the *trace* of A and $\det(I - A)$ denotes the *operator determinant* of $I - A$.

The following result was proved by Widom [26, Theorem 6.2] (see also [6, Section 10.90]). It extends earlier results by Szegő (see [11]) and now it is usually called the Szegő-Widom asymptotic trace formula.

Theorem 1.1 (Widom)**.** *Let $N \geq 1$. If a belongs to $K_{N \times N}^{1/2,1/2}$ and f is analytic on* $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$, *then*

$$(1.1) \qquad \operatorname{trace} f(T_n(a)) = (n+1)G_f(a) + E_f(a) + o(1) \quad as \ \ n \to \infty,$$

where

$$G_f(a) := \frac{1}{2\pi} \int_0^{2\pi} (\operatorname{trace} f(a))(e^{i\theta})d\theta,$$

$$E_f(a) := \frac{1}{2\pi i} \int_{\partial \Omega} f(\lambda) \frac{d}{d\lambda} \log \det T[a - \lambda]T[(a - \lambda)^{-1}]d\lambda,$$

and Ω is any bounded open set containing $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$ *on the closure of which f is analytic.*

Our main result is the following refinement of Theorem 1.1, which gives a higher order asymptotic trace formula.

Theorem 1.2. *Let $N \geq 1$ and $\alpha, \beta \in (0,1)$. Suppose that $\alpha + \beta \geq 1$. If $a \in K_{N \times N}^{\alpha,\beta}$ and f is analytic on* $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$, *then* (1.1) *is true with $o(1)$ replaced by $o(n^{1-\alpha-\beta})$.*

Notice that higher order asymptotic trace formulas are known for other classes of symbols: see [25] for $W \cap K^{\alpha,\alpha}$ with $\alpha > 1/2$ (here W stands for the Wiener algebra of functions with absolutely convergent Fourier series), [14] for weighted Wiener algebras, [15] for Hölder-Zygmund spaces, [16] for generalized Hölder spaces. All these classes consist of continuous functions only. More precisely, they are decomposing algebras of continuous functions in the sense of Budyanu and Gohberg. An invertible matrix function in such an algebra admits a Wiener-Hopf factorization within the algebra. The proofs of [14, 15, 16] are based on a combination of this observation and an approach of Böttcher and Silbermann [3] (see also [4, Sections 6.15–6.22] and [6, Sections 10.34–10.40]) to higher order asymptotic formulas of Toeplitz determinants with Widom's original proof of Theorem 1.1 (see [26] and [6, Section 10.90]). As far as we know, Vasil'ev, Maximenko, and Simonenko have never published a proof of the result stated in the short note [25], however, their result can be proved by the same method.

Generalized Krein algebras $K^{\alpha,\beta}$ may contain discontinuous functions. To study them we need a more advanced factorization theory in decomposing algebras of L^∞ functions developed by Heinig and Silbermann [13]. We present main results of this theory in Section 2 and then apply them to $K^{\alpha,\beta}$ with $\alpha + \beta \geq 1$ and $\max\{\alpha, \beta\} > 1/2$. Under these assumptions, if both Toeplitz operators $T(a)$ and $T(\widetilde{a})$ are invertible, then a admits simultaneously canonical right and left Wiener-Hopf factorizations $a = u_- u_+ = v_+ v_-$ in $K_{N \times N}^{\alpha,\beta}$. The factors and their inverses in these factorizations are stable under small perturbations of a in the norm of $K_{N \times N}^{\alpha,\beta}$.

We will use this fact in Section 4 for factorizations of $a - \lambda$, where λ belongs to a compact neighborhood Σ of the boundary of a set Ω containing $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$.

Section 3 contains some preliminaries on trace class operators and their determinants. Further we formulate the Borodin-Okounkov formula under weakened smoothness assumptions. This is an exact formula which relates determinants of finite Toeplitz matrices $\det T_n(a)$ and operator determinants of $I - Q_n H(b) H(\widetilde{c}) Q_n$, where $Q_n H(b) H(\widetilde{c}) Q_n$ are truncations of the product of Hankel operators $H(b)$ and $H(\widetilde{c})$ with $b := v_- u_+^{-1}$ and $c := u_-^{-1} v_+$. Here $Q_n := I - P_n$ and P_n is the finite section projection.

If $a - \lambda \in K_{N \times N}^{\alpha, \beta}$, then we can effectively estimate the speed of convergence of the trace class norm of $I - Q_n H[b(\lambda)] H[\widetilde{c(\lambda)}] Q_n$ to zero as $n \to \infty$ uniformly in $\lambda \in \Sigma$. This speed is $o(n^{1-\alpha-\beta})$. Combining this estimate with the Borodin-Okounkov formula for $a - \lambda$ and then applying Widom's "differentiate-multiply-integrate" arguments with respect to $\lambda \in \Sigma$, we prove Theorem 1.2 in Section 4.

2. Wiener-Hopf factorization and generalized Krein algebras

2.1. **Wiener-Hopf factorization in decomposing algebras.** For a unital algebra A, let $\mathcal{G} A$ denote the its group of invertible elements.

Mark Krein [17] was the first to understand the Banach algebraic background of Wiener-Hopf factorization and to present the method in a crystal-clear manner. Gohberg and Krein [9] proved that $a \in \mathcal{G} W_{N \times N}$ admits a Wiener-Hopf factorization. Later Budyanu and Gohberg developed an abstract factorization theory in decomposing algebras of *continuous* functions. Their results are contained in [7, Chap. 2]. Heinig and Silbermann [13] extended the theory of Budyanu and Gohberg to the case of decomposing algebras which may contain *discontinuous* functions. The following definitions and results are taken from [13] (see also [4, Chap. 5]).

Let A be a Banach algebra of complex-valued functions on the unit circle $\mathbb{T}$ under a Banach algebra norm $\| \cdot \|_A$. The algebra A is said to be *decomposing* if it possesses the following properties:

(a) A is continuously embedded in L^∞;
(b) A contains all Laurent polynomials;
(c) $PA \subset A$ and $QA \subset A$.

Using the closed graph theorem it is easy to deduce from (a)–(c) that P and Q are bounded on A and that PA and QA are closed subalgebras of A. For $k \in \mathbb{Z}$ and $t \in \mathbb{T}$, put $\chi_k(t) := t^k$. Given a decomposing algebra A put

$$A_+ = PA, \quad \overset{\circ}{A}_- = QA, \quad \overset{\circ}{A}_+ = \chi_1 A_+, \quad A_- = \chi_1 \overset{\circ}{A}_- .$$

Let A be a decomposing algebra. A matrix function $a \in A_{N \times N}$ is said to *admit a right* (resp. *left*) *Wiener-Hopf factorization in* $A_{N \times N}$ if it can be represented in the form

$$a = a_- d a_+ \quad (\text{resp. } a = a_+ d a_-),$$

where $a_\pm \in \mathcal{G}(A_\pm)_{N \times N}$ and

$$d = \operatorname{diag}(\chi_{\kappa_1}, \ldots, \chi_{\kappa_N}), \quad \kappa_i \in \mathbb{Z}, \quad \kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_N.$$

The integers κ_i are usually called the *right* (resp. *left*) *partial indices* of a; they can be shown to be uniquely determined by a. If $\kappa_1 = \cdots = \kappa_N = 0$, then the Wiener-Hopf factorization is said to be *canonical*. A decomposing algebra A is said

to have the *factorization property* if every matrix function in $\mathcal{G}A_{N \times N}$ admits a right Wiener-Hopf factorization in $A_{N \times N}$.

Let $\mathcal{R}$ be the restriction to the unit circle $\mathbb{T}$ of the set of all rational functions defined on the whole plane $\mathbb{C}$ and having no poles on $\mathbb{T}$.

Theorem 2.1. *Let A be a decomposing algebra. If at least one of the sets*

$$(\mathcal{R} \cap \overset{\circ}{A}_-) + A_+ \quad or \quad \overset{\circ}{A}_- + (\mathcal{R} \cap A_+)$$

is dense in A, then A has the factorization property.

2.2. **Stability of factors and their inverses under small perturbations.** Let A be a Banach algebra equipped with a norm $\| \cdot \|_A$. We will always consider an admissible norm $\| \cdot \|_{A_{N \times N}}$ in $A_{N \times N}$. Recall that a Banach algebra norm is said to be admissible (see [6, Section 1.29]) if there exist positive constants m and M such that

$$m \max_{1 \leq i,j \leq N} \|a_{ij}\|_A \leq \|a\|_{A_{N \times N}} \leq M \max_{1 \leq i,j \leq N} \|a_{ij}\|_A$$

for every matrix $a = (a_{ij})_{i,j=1}^N \in A_{N \times N}$.

The following result can be extracted from a stability theorem for factors and their inverses in the Wiener-Hopf factorization in decomposing algebras given in [20, Theorem 6.15]. There it was assumed, in addition, that a decomposing algebra is continuously embedded in the set of all continuous functions. However, the result is also true for decomposing algebras in the sense of Heinig and Silbermann adopted in this paper.

Theorem 2.2. *Let A be a decomposing algebra and $N \geq 1$. Suppose $a, c \in A_{N \times N}$ both admit canonical right (resp. left) Wiener-Hopf factorizations in $A_{N \times N}$. Then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if*

$$\|a - c\|_{A_{N \times N}} < \delta,$$

then for every canonical right (resp. left) Wiener-Hopf factorization $a = a_-^{(r)} a_+^{(r)}$ (resp. $a = a_+^{(l)} a_-^{(l)}$) one can choose a canonical right (resp. left) Wiener-Hopf factorization $c = c_-^{(r)} c_+^{(r)}$ (resp. $c = c_+^{(l)} c_-^{(l)}$) such that

$$\|a_\pm^{(r)} - c_\pm^{(r)}\|_{A_{N \times N}} < \varepsilon, \quad \|[a_\pm^{(r)}]^{-1} - [c_\pm^{(r)}]^{-1}\|_{A_{N \times N}} < \varepsilon$$

$$(resp. \quad \|a_\pm^{(l)} - c_\pm^{(l)}\|_{A_{N \times N}} < \varepsilon, \quad \|[a_\pm^{(l)}]^{-1} - [c_\pm^{(l)}]^{-1}\|_{A_{N \times N}} < \varepsilon).$$

2.3. **Invertibility in generalized Krein algebras.** For $1 \leq p \leq \infty$, let $\overline{H^p} := \{a \in L^p : \overline{a} \in H^p\}$ and let $C := C(\mathbb{T})$ denote the set of all continuous functions on $\mathbb{T}$. If $\alpha, \beta \geq 1/2$, then in view of [2, Lemma 6.2],

$$(2.1) \qquad\qquad K^{\alpha,0} \subset C + H^\infty, \quad K^{0,\beta} \subset C + \overline{H^\infty}.$$

Hence, if $\alpha, \beta \in (0,1)$ and $\alpha + \beta \geq 1$, then

$$(2.2) \qquad\qquad K^{\alpha,\beta} \subset (C + H^\infty) \cup (C + \overline{H^\infty}).$$

The following result was proved by Krein [18] for $\alpha = \beta = 1/2$.

Theorem 2.3 (see [2, Theorem 1.4]). *Let $\alpha, \beta \in (0,1)$.*

(a) *Suppose $\alpha \geq 1/2$ and K is either $K^{\alpha,0}$ or $K^{\alpha,1-\alpha}$. If $a \in K$, then*

$$a \in \mathcal{G}K \Longleftrightarrow a \in \mathcal{G}(C + H^\infty).$$

(b) *Suppose $\beta \geq 1/2$ and K is either $K^{0,\beta}$ or $K^{1-\beta,\beta}$. If $a \in K$, then*
$$a \in \mathcal{G}K \Longleftrightarrow a \in \mathcal{G}(C + \overline{H^\infty}).$$

Corollary 2.4. *Let $\alpha, \beta \in (0,1)$.*
   (a) *Suppose $\alpha \geq 1 - \beta \geq 1/2$. If $a \in K^{\alpha,\beta}$, then*
$$a \in \mathcal{G}K^{\alpha,\beta} \Longleftrightarrow a \in \mathcal{G}(C + H^\infty).$$
   (b) *Suppose $\beta \geq 1 - \alpha \geq 1/2$. If $a \in K^{\alpha,\beta}$, then*
$$a \in \mathcal{G}K^{\alpha,\beta} \Longleftrightarrow a \in \mathcal{G}(C + \overline{H^\infty}).$$
   (c) *Suppose $\alpha \geq \beta \geq 1/2$ or $\beta \geq \alpha \geq 1/2$. If $a \in K^{\alpha,\beta}$, then*
$$a \in \mathcal{G}K^{\alpha,\beta} \Longleftrightarrow a \in \mathcal{G}\big((C + H^\infty) \cap (C + \overline{H^\infty})\big).$$

Proof. (a) Let $a \in K^{\alpha,\beta} = K^{1-\beta,\beta} \cap K^{\alpha,0}$. By Theorem 2.3(a),
$$a \in \mathcal{G}K^{1-\beta,\beta} \Longleftrightarrow a \in \mathcal{G}(C + H^\infty), \quad a \in \mathcal{G}K^{\alpha,0} \Longleftrightarrow a \in \mathcal{G}(C + H^\infty).$$
Thus $a \in \mathcal{G}K^{\alpha,\beta} \Longleftrightarrow a \in \mathcal{G}(C + H^\infty)$. Part (a) is proved. Part (b) follows form Theorem 2.3(b) in the same fashion.

(c) Let $a \in K^{\alpha,\beta} = K^{\alpha,0} \cap K^{0,\beta}$. From Theorem 2.3 it follows that
$$a \in \mathcal{G}K^{\alpha,0} \Longleftrightarrow \mathcal{G}(C + H^\infty), \quad a \in \mathcal{G}K^{0,\beta} \Longleftrightarrow \mathcal{G}(C + \overline{H^\infty}).$$
Hence $a \in \mathcal{G}K^{\alpha,\beta} = G(K^{\alpha,0} \cap K^{0,\beta}) \Longleftrightarrow a \in \mathcal{G}\big((C + H^\infty) \cap (C + \overline{H^\infty})\big)$. Part (c) is proved. $\square$

2.4. **Wiener-Hopf factorization in generalized Krein algebras.**

Lemma 2.5. *Let $\alpha, \beta \in (0,1)$ and $\max\{\alpha, \beta\} > 1/2$. Then $K^{\alpha,\beta}$ is a decomposing algebra with the factorization property.*

Proof. The statement is proved by analogy with [2, Lemma 7.7]. By [2, Lemma 6.1], the projections P and Q are bounded on $K^{\alpha,\beta}$. Hence $K^{\alpha,\beta}$ is a decomposing algebra. Assume that $\beta > 1/2$. Taking into account that
$$K^{\alpha,\beta} = L^\infty \cap (QB_2^\alpha + PB_2^\beta),$$
where B_2^α and B_2^β are Besov spaces, from [22, Sections 3.5.1 and 3.5.5] one can deduce that $\mathcal{R} \cap PK^{\alpha,\beta}$ is dense in $PK^{\alpha,\beta}$. Analogously, if $\alpha > 1/2$, then $\mathcal{R} \cap QK^{\alpha,\beta}$ is dense in $QK^{\alpha,\beta}$. Theorem 2.1 gives the factorization property of $K^{\alpha,\beta}$. $\square$

Theorem 2.6. *Let $N \geq 1$, $\alpha, \beta \in (0,1)$, $\alpha + \beta \geq 1$, and $\max\{\alpha, \beta\} > 1/2$. If $a \in K_{N\times N}^{\alpha,\beta}$ and both $T(a)$ and $T(\widetilde{a})$ are invertible on H_N^2, then a is invertible in $K_{N\times N}^{\alpha,\beta}$ and admits canonical right and left Wiener-Hopf factorizations in $K_{N\times N}^{\alpha,\beta}$.*

Proof. Once one has at hands Corollary 2.4 and Lemma 2.5, the proof is developed as in [2, Theorem 1.7(a)]. For the convenience of the reader we give a complete proof here.

Suppose $\alpha = \max\{\alpha, \beta\}$. It is clear that for every $\beta \in (0,1)$ one has $\beta \geq 1/2$ or $1 - \beta \geq 1/2$. Thus $\alpha \geq \beta \geq 1/2 \ (\geq 1 - \beta)$ or $\alpha \geq 1 - \beta \geq 1/2 \ (\geq \beta)$. In the first case from (2.1) it follows that
$$K_{N\times N}^{\alpha,\beta} \subset (C + H^\infty)_{N\times N} \cap (C + \overline{H^\infty})_{N\times N}.$$
Since $T(\widetilde{a})$ and $T(a)$ are invertible, from [6, Theorem 2.94(a)] we deduce that $\det a$ and $\det \widetilde{a}$ belong to $\mathcal{G}(C + H^\infty)$. Hence, $\det a$ belongs to $\mathcal{G}\big((C + H^\infty) \cap (C + \overline{H^\infty})\big)$.

By Corollary 2.4(c), $\det a \in \mathcal{G}K^{\alpha,\beta}$. Then, in view of [19, Chap. 1, Theorem 1.1], $a \in \mathcal{G}K_{N\times N}^{\alpha,\beta}$.

The case $\alpha \geq 1 - \beta \geq 1/2$ is treated with the help of Corollary 2.4(a). Then $a \in \mathcal{G}K_{N\times N}^{\alpha,\beta}$. Analogously, if $\beta = \max\{\alpha,\beta\}$, then by using Corollary 2.4(b) or (c), one cans how that $a \in \mathcal{G}K_{N\times N}^{\alpha,\beta}$.

By Simonenko's factorization theorem (see, e.g. [7, Chap. 7, Theorem 3.2] or [20, Theorem 3.14]), if $T(a)$ is invertible on H_N^2, then a admits a canonical right generalized factorization in L_N^2, that is, there exist functions a_-, a_+ such that $a = a_- a_+$ and $a_-^{\pm 1} \in (\overline{H^2})_{N\times N}$, $a_+^{\pm 1} \in (H^2)_{N\times N}$. On the other hand, in view of Lemma 2.5, a admits a right Wiener-Hopf factorization in $K_{N\times N}^{\alpha,\beta}$, that is, there exist functions $u_\pm \in \mathcal{G}(K_\pm^{\alpha,\beta})_{N\times N}$ such that $a = u_- d u_+$ and d is a diagonal term of the form $d = \mathrm{diag}(\chi_{\kappa_1}, \ldots, \chi_{\kappa_N})$. It is clear that $u_-^{\pm 1} \in (\overline{H^2})_{N\times N}$ and $u_+^{\pm 1} \in (H^2)_{N\times N}$. Thus $a = u_- d u_+$ is a right generalized factorization of a in L_N^2. It is well known that the set of partial indices of such a factorization is unique (see, e.g. [20, Corollary 2.1]). Thus d is the identity matrix and $a = u_- u_+$.

Since $T(\widetilde{a})$ is invertible on H_N^2, from [6, Proposition 7.19(b)] it follows that $T(a^{-1})$ is also invertible on H_N^2. By what has just been proved, there exist functions $f_\pm \in \mathcal{G}(K_\pm^{\alpha,\beta})_{N\times N}$ such that $a^{-1} = f_- f_+$. Put $v_\pm := f_\pm^{-1}$. Then $a = v_+ v_-$ and $v_\pm \in \mathcal{G}(K_\pm^{\alpha,\beta})_{N\times N}$. $\qquad\square$

3. The Borodin-Okounkov formula

3.1. Trace class operators, Hilbert-Schmidt operators, and operator determinants.
In this subsection we collect necessary facts from general operator theory in Hilbert spaces (see [10, Chap. 3–4]).

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. For a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ and $n \in \mathbb{Z}_+$, we define

$$s_n(A) := \inf\big\{\|A - F\| : \dim F \leq n\big\}.$$

For $1 \leq p < \infty$, the collection $\mathcal{C}_p(\mathcal{H},\mathcal{K})$ of all bounded linear operators $A : \mathcal{H} \to \mathcal{K}$ satisfying

$$\|A\|_{\mathcal{C}_p(\mathcal{H},\mathcal{K})} := \left(\sum_{n=0}^{\infty} s_n^p(A)\right)^{1/p} < \infty$$

is called the p-Schatten-von Neumann class. If $p = 1$, then $\mathcal{C}_1(\mathcal{H},\mathcal{K})$ is called the *trace class* and if $p = 2$, then $\mathcal{C}_2(\mathcal{H},\mathcal{K})$ is called the class of Hilbert-Schmidt operators. We will simply write $\mathcal{C}_p(\mathcal{H})$ instead of $\mathcal{C}_p(\mathcal{H},\mathcal{H})$.

One can show that, for every $A \in \mathcal{C}_1(\mathcal{H})$ and for every orthonormal basis $\{\varphi_j\}_{j=0}^\infty$ of $\mathcal{H}$, the series $\sum_{j=0}^\infty \langle A\varphi_j, \varphi_j\rangle_\mathcal{H}$ converges absolutely and that its sum does not depend on the particular choice of $\{\varphi_j\}_{j=0}^\infty$. This sum is denoted by $\mathrm{trace}\, A$ and is referred to as the *trace* of A. It is well known that

$$|\,\mathrm{trace}\, A| \leq \|A\|_{\mathcal{C}_1(\mathcal{H})}$$

for all $A \in \mathcal{C}_1(\mathcal{H})$.

The Hilbert-Schmidt norm of an operator $A \in \mathcal{C}_2(\mathcal{H}, \mathcal{K})$ can be expressed in the form

$$\|A\|_{\mathcal{C}_2(\mathcal{H},\mathcal{K})} = \left(\sum_{j,k}^{\infty} |\langle A\varphi_j, \psi_k \rangle_{\mathcal{K}}|^2 \right)^{1/2},$$

where $\{\varphi_j\}_{j=0}^{\infty}$ and $\{\psi_k\}_{k=0}^{\infty}$ are orthonormal bases of $\mathcal{H}$ and $\mathcal{K}$, respectively.

We will need the following version of the Hölder inequality. If $B \in \mathcal{C}_2(\mathcal{H}, \mathcal{K})$ and $A \in \mathcal{C}_2(\mathcal{K}, \mathcal{H})$, then $AB \in \mathcal{C}_1(\mathcal{H})$ and

$$\|AB\|_{\mathcal{C}_1(\mathcal{H})} \leq \|A\|_{\mathcal{C}_2(\mathcal{K},\mathcal{H})} \|B\|_{\mathcal{C}_2(\mathcal{H},\mathcal{K})}. \tag{3.1}$$

Let A be a bounded linear operator on $\mathcal{H}$ of the form $I + K$ with $K \in \mathcal{C}_1(\mathcal{H})$. If $\{\lambda_j(K)\}_{j \geq 0}$ denotes the sequence of the nonzero eigenvalues of K (counted up to algebraic multiplicity), then $\sum_{j=0}^{\infty} |\lambda_j(K)| < \infty$. Therefore the (possibly infinite) product $\prod_{j \geq 0} (1 + \lambda_j(K))$ is absolutely convergent. The *operator determinant* of A is defined by

$$\det A = \det(I + K) = \prod_{j \geq 0} (1 + \lambda_j(K)).$$

In the case where the spectrum of K consists only of 0 we put $\det(I + K) = 1$.

Lemma 3.1. *If $A \in \mathcal{C}_1(\mathcal{H})$ and $\|A\|_{\mathcal{C}_1(\mathcal{H})} < 1$, then $|\log \det(I - A)| \leq 2\|A\|_{\mathcal{C}_1(\mathcal{H})}$.*

Proof. Since $A \in \mathcal{C}_1(\mathcal{H})$, by formula (1.16) of [10, Chap. IV],

$$\log \det(I - A) = \text{trace} \log(I - A). \tag{3.2}$$

On the other hand,

$$\log(I - A) = -\sum_{j=1}^{\infty} \frac{1}{j} A^j. \tag{3.3}$$

From (3.2), (3.3), and $|\text{trace } A| \leq \|A\|_{\mathcal{C}_1(\mathcal{H})}$ we get

$$|\log \det(I - A)| \leq \left| \text{trace} \left[\sum_{j=1}^{\infty} \frac{1}{j} A^j \right] \right| \leq \sum_{j=1}^{\infty} |\text{trace } A^j| \leq \sum_{j=1}^{\infty} \|A^j\|_{\mathcal{C}_1(\mathcal{H})}. \tag{3.4}$$

By Hölder's inequality,

$$\|A^j\|_{\mathcal{C}_1(\mathcal{H})} \leq \|A\|_{\mathcal{C}_1(\mathcal{H})} \|A^{j-1}\|_{\mathcal{C}_\infty(\mathcal{H})} \leq \|A\|_{\mathcal{C}_1(\mathcal{H})} \|A\|_{\mathcal{C}_\infty(\mathcal{H})}^{j-1}.$$

Taking into account that $\|A\|_{\mathcal{C}_\infty(\mathcal{H})} \leq \|A\|_{\mathcal{C}_1(\mathcal{H})}$, we get $\|A^j\|_{\mathcal{C}_1(\mathcal{H})} \leq \|A\|_{\mathcal{C}_1(\mathcal{H})}^j$. Hence, (3.4) yields

$$|\log \det(I - A)| \leq \sum_{j=1}^{\infty} \|A\|_{\mathcal{C}_1(\mathcal{H})}^j = \frac{\|A\|_{\mathcal{C}_1(\mathcal{H})}}{1 - \|A\|_{\mathcal{C}_1(\mathcal{H})}} \leq 2\|A\|_{\mathcal{C}_1(\mathcal{H})}$$

because $\|A\|_{\mathcal{C}_1(\mathcal{H})} < 1$. $\qquad\square$

3.2. The Borodin-Okounkov formula under weakened hypotheses.
For $a \in L_{N \times N}^\infty$ and $n \in \mathbb{Z}_+$, define the operators

$$P_n : \sum_{k=0}^\infty \widehat{a}(k)\chi_k \mapsto \sum_{k=0}^n \widehat{a}(k)\chi_k, \quad Q_n := I - P_n.$$

The operator $P_n T(a) P_n : P_n H_N^2 \to P_n H_N^2$ may be identified with the finite block Toeplitz matrix $T_n(a) = (\widehat{a}(j-k))_{j,k=0}^n$.

In June 1999, Its and Deift raised the question whether there is a general formula that expresses the determinant of the Toeplitz matrix $T_n(a)$ as the operator determinant of an operator $I - K$ where K acts on $\ell_2\{n+1, n+2, \dots\}$. Borodin and Okounkov showed in 2000 that such a formula exists (however, it was known even much earlier. In 1979, Geronimo and Case used it to prove the strong Szegő limit theorem). Further, in 2000, several different proofs of it were found by Basor and Widom and by Böttcher. We refer to the books by Simon [24], Böttcher and Grudsky [1, Section 2.8], Böttcher and Silbermann [6, Section 10.40] for the exact references, proofs, and historical remarks on this beautiful piece of mathematics. Below we formulate the Borodin-Okounkov formula in a form suggested by Widom under assumptions on the symbol a of $T_n(a)$ which are slightly weaker than in [6, Section 10.40].

Theorem 3.2. *Suppose $a \in (C + H^\infty)_{N \times N} \cup (C + \overline{H^\infty})_{N \times N}$ satisfies the following hypothesis:*

    (i) *there are two factorizations $a = u_- u_+ = v_+ v_-$, where $u_-, v_- \in \mathcal{G}(\overline{H^\infty})_{N \times N}$ and $u_+, v_+ \in \mathcal{G}(H^\infty)_{N \times N}$;*

Put $b := v_- u_+^{-1}$ and $c := u_-^{-1} v_+$. Suppose that

    (ii) $H(b)H(\widetilde{c}) \in \mathcal{C}_1(H_N^2)$.

Then the constants

(3.5)
$$G(a) := \lim_{r \to 1-0} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \det a_r(e^{i\theta}) d\theta\right),$$

where

$$a_r(e^{i\theta}) := \sum_{n=-\infty}^\infty \widehat{a}(n) r^{|n|} e^{in\theta},$$

and

$$E(a) := \frac{1}{\det T(b)T(c)}$$

are well defined, are not equal to zero, and the Borodin-Okounkov formula

(3.6)
$$\det T_n(a) = G(a)^{n+1} E(a) \det\left(I - Q_n H(b)H(\widetilde{c})Q_n\right)$$

holds for every $n \in \mathbb{N}$. If, in addition,

    (iii) $H(a)H(\widetilde{a}^{-1}) \in \mathcal{C}_1(H_N^2)$,

then $E(a) = \det T(a)T(a^{-1})$.

Proof. From (i) and [6, Proposition 2.14] it follows that the operators $T(a)$ and $T(a^{-1})$ are invertible on H_N^2. If $a \in (C + H^\infty)_{N \times N}$, then from [6, Proposition 10.6(a)] we deduce that the limit in (3.5) exists, is finite and nonzero. Hence the constant $G(a)$ is well defined. The proof of [6, Proposition 10.6(a)] equally works also for the case $a \in (C + \overline{H^\infty})_{N \times N}$. Therefore, $G(a)$ is well defined in this case as well.

From [6, Proposition 2.14] it follows also that the operators $T(b)$, $T(c)$ are invertible and $T(b)T(c) = I - H(b)H(\widetilde{c})$. From (ii) and the above equality we get that $\det T(b)T(c)$ makes sense. Since $T(b)T(c)$ is invertible, $\det T(b)T(c) \neq 0$ and therefore the constant $E(a)$ is well defined.

The Borodin-Okounkov formula (3.6) is proved in [6, Section 10.40] under the assumption that $a \in K_{N \times N}^{1/2, 1/2}$ admits right and left canonical Wiener-Hopf factorizations in $K_{N \times N}^{1/2, 1/2}$. The two proofs given in [6, Section 10.40] work equally under weaker hypotheses (i)–(ii).

Applying [6, Proposition 2.14], we get

$$I - H(b)H(\widetilde{c}) = T(b)T(c) = T(v_-)T(u_+^{-1})T(u_-^{-1})T(v_+)$$

and

$$
\begin{aligned}
T(v_+)\big(I - H(b)H(\widetilde{c})\big)T^{-1}(v_+) &= I - T(v_+)H(b)H(\widetilde{c})T^{-1}(v_+) \\
&= T(v_+)T(v_-)T(u_+^{-1})T(u_-^{-1}) \\
&= T^{-1}(a^{-1})T^{-1}(a).
\end{aligned}
$$

From these equalities and [10, Chap. IV, Section 1.6] it follows that

$$
\begin{aligned}
(3.7) \qquad \det T(b)T(c) &= \det\big(I - H(b)H(\widetilde{c})\big) \\
&= \det\big(I - T(v_+)H(b)H(\widetilde{c})T^{-1}(v_+)\big) \\
&= \det T^{-1}(a^{-1})T^{-1}(a).
\end{aligned}
$$

From (iii) and $T(a)T(a^{-1}) = I - H(a)H(\widetilde{a}^{-1})$ it follows that $\det T(a)T(a^{-1})$ makes sense. By [10, Chap. IV, Section 1.7],

$$\det T^{-1}(a^{-1})T^{-1}(a) \cdot \det T(a)T(a^{-1}) = \det T^{-1}(a^{-1})T^{-1}(a)T(a)T(a^{-1}) = 1.$$

Hence

$$(3.8) \qquad \det T^{-1}(a^{-1})T^{-1}(a) = \frac{1}{\det T(a)T(a^{-1})}.$$

Combining (3.7) and (3.8), we arrive at $E(a) = \det T(a)T(a^{-1})$. $\square$

4. Proof of the main result

4.1. Hilbert-Schmidt norms of truncations of Hankel operators. Let $\gamma \in \mathbb{R}$. By ℓ_2^γ we denote the Hilbert space of all sequences $\{\varphi_k\}_{k=0}^\infty$ such that

$$\sum_{k=0}^\infty |\varphi_k|^2 (k+1)^{2\gamma} < \infty.$$

Clearly, the sequence $\{e_k/(k+1)^\gamma\}_{k=0}^\infty$, where $(e_k)_j = \delta_{kj}$ and δ_{kj} is the Kronecker delta, is an orthonormal basis of ℓ_2^γ. If $\gamma = 0$, we will simply write ℓ_2 instead of ℓ_2^0.

In this subsection we will estimate Hilbert-Schmidt norms of truncations of Hankel operators acting between ℓ_2 and ℓ_2^γ by the rules

$$H(a) : \{\varphi_j\}_{j=0}^\infty \mapsto \Big\{ \sum_{j=0}^\infty \widehat{a}(i+j+1)\varphi_j \Big\}_{i=0}^\infty,$$

$$H(\widetilde{a}) : \{\varphi_j\}_{j=0}^\infty \mapsto \Big\{ \sum_{j=0}^\infty \widehat{a}(-i-j-1)\varphi_j \Big\}_{i=0}^\infty.$$

Notice that one can identify Hankel operators acting on H^2 and on ℓ^2. For $\varphi = \{\varphi_j\}_{j=0}^\infty$ and $n \in \mathbb{Z}_+$, define

$$(Q_n\varphi)_j = \begin{cases} \varphi_j & \text{if } j \geq n+1, \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in K^{\alpha,\beta}$ and $n \in \mathbb{N}$, put

$$r_n^-(a) := \left(\sum_{k=n+1}^\infty |\widehat{a}(-k)|^2 (k+1)^{2\alpha} \right)^{1/2},$$

$$r_n^+(a) := \left(\sum_{k=n+1}^\infty |\widehat{a}(k)|^2 (k+1)^{2\beta} \right)^{1/2}.$$

Lemma 4.1. *Let* $-1/2 < \gamma < 1/2$ *and* $\alpha, \beta \in (0,1)$. *Suppose* $b, c \in K^{\alpha,\beta}$.

(a) *If* $\alpha \geq \gamma + 1/2$, *then there exists a positive constant* $M(\alpha, \gamma)$ *depending only on* α *and* γ *such that for all sufficiently large* n,

(4.1) $$\|H(\widetilde{c})Q_n\|_{\mathcal{C}_2(\ell_2, \ell_2^\gamma)} \leq M(\alpha, \gamma) \frac{r_{n+1}^-(c)}{n^{\alpha - \gamma - 1/2}}.$$

(b) *If* $\beta \geq -\gamma + 1/2$, *then there exists a positive constant* $M(\beta, \gamma)$ *depending only on* β *and* γ *such that for all sufficiently large* n,

(4.2) $$\|Q_nH(b)\|_{\mathcal{C}_2(\ell_2^\gamma, \ell_2)} \leq M(\beta, \gamma) \frac{r_{n+1}^+(b)}{n^{\beta + \gamma - 1/2}}.$$

Proof. (a) It is easy to see that

$$(H(\widetilde{c})Q_ne_j)_k = \begin{cases} \widehat{c}(-k-j-1) & \text{if } j \geq n+1, \ k \in \mathbb{Z}_+, \\ 0 & \text{otherwise.} \end{cases}$$

Then

(4.3)
$$\begin{aligned}
\|H(\widetilde{c})Q_n\|_{\mathcal{C}_2(\ell_2, \ell_2^\gamma)}^2 &= \sum_{j,k=0}^\infty \left| \left\langle H(\widetilde{c})Q_ne_j, \frac{e_k}{(k+1)^\gamma} \right\rangle_{\ell_2^\gamma} \right|^2 \\
&= \sum_{j,k=0}^\infty |(H(\widetilde{c})Q_ne_j)_k|^2 (k+1)^{2\gamma} \\
&= \sum_{k=0}^\infty \sum_{j=n+1}^\infty |\widehat{c}(-k-j-1)|^2 (k+1)^{2\gamma} \\
&= \sum_{k=n+2}^\infty |\widehat{c}(-k)|^2 \sum_{j=1}^{k-n-1} j^{2\gamma}.
\end{aligned}$$

If $-1/2 < \gamma < 1/2$, then

(4.4) $$\sum_{j=1}^{k-n-1} j^{2\gamma} \leq \frac{(k-n)^{1+2\gamma}}{1+2\gamma}.$$

From (4.3) and (4.4) it follows that

$$\|H(\widetilde{c})Q_n\|^2_{\mathcal{C}_2(\ell_2,\ell_2^\gamma)} \leq \frac{1}{1+2\gamma} \sum_{k=n+2}^{\infty} |\widehat{c}(-k)|^2 (k-n)^{1+2\gamma}$$

$$\text{(4.5)} \qquad \leq \frac{1}{1+2\gamma} \sum_{k=n+2}^{\infty} |\widehat{c}(-k)|^2 (k+1)^{2\alpha} \frac{(k-n)^{1+2\gamma}}{k^{2\alpha}}$$

$$\leq \frac{1}{1+2\gamma} \left(\sup_{k\geq n+2} \frac{(k-n)^{1+2\gamma}}{k^{2\alpha}} \right) \left[r_{n+1}^-(c) \right]^2.$$

If $1 + 2\gamma = 2\alpha$, then

$$\text{(4.6)} \qquad \sup_{k\geq n+2} \left(\frac{k-n}{k} \right)^{2\alpha} \leq 1.$$

Combining (4.5) and (4.6), we arrive at (4.1) with $M(\alpha,\gamma) = (2\alpha)^{-1/2}$.

If $\alpha > \gamma + 1/2$, then put

$$A := \frac{2\gamma+1}{2\alpha-2\gamma-1} > 0.$$

Let $n \geq 2/A$. Then

$$x_n := (A+1)n = \frac{2\alpha n}{2\alpha-2\gamma-1} \in [n+2,\infty).$$

It is not difficult to show that the function

$$f_n(x) := (x-n)^{1+2\gamma} x^{-2\alpha}, \quad x \in [n+2,\infty)$$

attains its absolute maximum at x_n. Thus

$$\text{(4.7)} \qquad \sup_{k\geq n+2} \frac{(k-n)^{1+2\gamma}}{k^{2\alpha}} \leq f_n(x_n) = \frac{A^{1+2\gamma}}{(A+1)^{2\alpha}} n^{1+2\gamma-2\alpha}.$$

Combining (4.5) and (4.7), we arrive at (4.1) with

$$M(\alpha,\gamma) := (1+2\gamma)^{-1/2} A^{1/2+\gamma} (A+1)^{-\alpha}$$

for all $n \geq 2/A$. Part (a) is proved. The proof of part (b) is analogous. $\square$

4.2. **Trace class norms of truncations of products of two Hankel operators.** The following fact is well known (see e.g. [6, Section 10.12] and also [23], [21, Chap. 6]).

Lemma 4.2. *Let* $N \geq 1$, $\alpha, \beta \in (0,1)$, *and* $\alpha + \beta \geq 1$. *If* $b,c \in K_{N\times N}^{\alpha,\beta}$, *then* $H(b)H(\widetilde{c}) \in \mathcal{C}_1(H_N^2)$.

We will also need a quantitative version of the above result for truncations of the product $H(b)H(\widetilde{c})$.

For $a \in K_{N\times N}^{\alpha,\beta}$ and $n \in \mathbb{N}$, put

$$R_n^-(a) := \left(\sum_{k=n+1}^{\infty} \|\widehat{a}(-k)\|^2_{\mathbb{C}^{N\times N}} (k+1)^{2\alpha} \right)^{1/2},$$

$$R_n^+(a) := \left(\sum_{k=n+1}^{\infty} \|\widehat{a}(k)\|^2_{\mathbb{C}^{N\times N}} (k+1)^{2\beta} \right)^{1/2}.$$

Lemma 4.3. *Let $N \geq 1$, $\alpha, \beta \in (0,1)$, and $\alpha + \beta \geq 1$. Then there exists a constant $L = L_{\alpha,\beta,N}$ depending only on N and α, β such that for every $b, c \in K_{N \times N}^{\alpha,\beta}$ and all sufficiently large n,*

$$(4.8) \qquad \|Q_n H(b) H(\widetilde{c}) Q_n\|_{\mathcal{C}_1(H_N^2)} \leq \frac{L}{n^{\alpha+\beta-1}} R_{n+1}^+(b) R_{n+1}^-(c).$$

Proof. Put $\gamma := 1/2 - \beta$. Then $\gamma \in (-1/2, 1/2)$ and $\alpha \geq \gamma + 1/2$ because $\alpha + \beta \geq 1$. Let b_{ij} and c_{ij}, where $i, j \in \{1, \ldots, N\}$, be the entries of $b, c \in K_{N \times N}^{\alpha,\beta}$, respectively. By Lemma 4.1, there exist positive constants $M(\alpha, \gamma)$ and $M(\beta, \gamma)$ depending only on α, β, and γ such that for all sufficiently large n and all $i, j \in \{1, \ldots, N\}$,

$$(4.9) \qquad \|Q_n H(b_{ij})\|_{\mathcal{C}_2(\ell_2^\gamma, \ell_2)} \leq M(\beta, \gamma) \frac{r_{n+1}^+(b_{ij})}{n^{\beta+\gamma-1/2}},$$

$$(4.10) \qquad \|H(\widetilde{c_{ij}}) Q_n\|_{\mathcal{C}_2(\ell_2, \ell_2^\gamma)} \leq M(\alpha, \gamma) \frac{r_{n+1}^-(c_{ij})}{n^{\alpha-\gamma-1/2}}.$$

From Hölder's inequality (3.1) and (4.9)–(4.10) it follows that

$$\|Q_n H(b_{ij}) H(\widetilde{c_{ij}}) Q_n\|_{\mathcal{C}_1(\ell_2)} \leq \|Q_n H(b_{ij})\|_{\mathcal{C}_2(\ell_2^\gamma, \ell_2)} \|H(\widetilde{c_{ij}}) Q_n\|_{\mathcal{C}_2(\ell_2, \ell_2^\gamma)}$$

$$(4.11) \qquad\qquad\qquad \leq M(\alpha, \gamma) M(\beta, \gamma) \frac{r_{n+1}^+(b_{ij}) r_{n+1}^-(c_{ij})}{n^{\alpha+\beta-1}}$$

for all $i, j \in \{1, \ldots, N\}$ and all large n.

It is not difficult to verify that there exist positive constants A_N and B_N depending only on the dimension N such that

$$(4.12) \qquad \|Q_n H(b) H(\widetilde{c}) Q_n\|_{\mathcal{C}_1(H_N^2)} \leq A_N \max_{1 \leq i,j \leq N} \|Q_n H(b_{ij}) H(\widetilde{c_{ij}}) Q_n\|_{\mathcal{C}_1(\ell_2)}$$

and

$$(4.13) \qquad \max_{1 \leq i,j \leq N} r_{n+1}^+(b_{ij}) \leq B_N R_{n+1}^+(b), \quad \max_{1 \leq i,j \leq N} r_{n+1}^-(c_{ij}) \leq B_N R_{n+1}^-(c)$$

for all sufficiently large n and all $b, c \in K_{N \times N}^{\alpha,\beta}$. Combining (4.11)–(4.13), we arrive at (4.8) with $L = L_{\alpha,\beta,N} := A_N B_N^2 M(\alpha, \gamma) M(\beta, \gamma)$. $\qquad\square$

4.3. **Tails of the norms of functions in generalized Krein algebras.**

Lemma 4.4. *Let $N \geq 1$, $\alpha, \beta \in (0,1)$, and $\max\{\alpha, \beta\} \geq 1/2$. Suppose Σ is a compact set in the complex plane. If $a : \Sigma \to K_{N \times N}^{\alpha,\beta}$ is a continuous function, then*

$$(4.14) \qquad \lim_{n \to \infty} \sup_{\lambda \in \Sigma} R_n^-\big(a(\lambda)\big) = 0, \quad \lim_{n \to \infty} \sup_{\lambda \in \Sigma} R_n^+\big(a(\lambda)\big) = 0.$$

Proof. This statement is proved by analogy with [14, Proposition 2.3] and [16, Lemma 6.2]. Let us prove the first equality in (4.14). Assume the contrary. Then there exist a constant $C > 0$ and a sequence $\{n_k\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} \sup_{\lambda \in \Sigma} R_{n_k}^-\big(a(\lambda)\big) \geq C.$$

Hence there are a number $k_0 \in \mathbb{N}$ and a sequence $\{\lambda_k\}_{k=k_0}^\infty$ such that for all $k \geq k_0$,

$$(4.15) \qquad R_{n_k}^-\big(a(\lambda_k)\big) \geq \frac{C}{2} > 0.$$

Since $\{\lambda_k\}_{k=k_0}^\infty$ is bounded, there is its convergent subsequence $\{\lambda_{k_j}\}_{j=1}^\infty$. Let λ_0 be the limit of this subsequence. Clearly, $\lambda_0 \in \Sigma$ because Σ is closed. Since the function $a : \Sigma \to K_{N \times N}^{\alpha,\beta}$ is continuous at λ_0, for every $\varepsilon \in (0, C/2)$, there exists a

$\Delta > 0$ such that $|\lambda - \lambda_0| < \Delta, \lambda \in \Sigma$ implies $\|a(\lambda) - a(\lambda_0)\|_{K^{\alpha,\beta}_{N \times N}} < \varepsilon$. Because $\lambda_{k_j} \to \lambda_0$ as $j \to \infty$, for that Δ there is a number $J \in \mathbb{N}$ such that $|\lambda_{k_j} - \lambda_0| < \Delta$ for all $j \geq J$, and thus

$$(4.16) \qquad \|a(\lambda_{k_j}) - a(\lambda_0)\|_{K^{\alpha,\beta}_{N \times N}} < \varepsilon \quad \text{for all} \quad j \geq J.$$

On the other hand, (4.15) implies that

$$(4.17) \qquad R^-_{n_{k_j}}\big(a(\lambda_{k_j})\big) \geq \frac{C}{2} > 0 \quad \text{for all} \quad j \geq J.$$

By the Minkowski inequality,

$$(4.18) \qquad \begin{aligned} R^-_{n_{k_j}}\big(a(\lambda_{k_j})\big) &\leq R^-_{n_{k_j}}\big(a(\lambda_0)\big) + R^-_{n_{k_j}}\big(a(\lambda_{k_j}) - a(\lambda_0)\big) \\ &\leq R^-_{n_{k_j}}\big(a(\lambda_0)\big) + \|a(\lambda_{k_j}) - a(\lambda_0)\|_{K^{\alpha,\beta}_{N \times N}}. \end{aligned}$$

From (4.16)–(4.18) we get for all $j \geq J$,

$$R^-_{n_{k_j}}\big(a(\lambda_0)\big) \geq \frac{C}{2} - \varepsilon > 0.$$

Therefore,

$$\sum_{k=0}^{\infty} \big\|[a(\lambda_0)]^\widehat{}(-k)\big\|^2_{\mathbb{C}_{N \times N}} (k+1)^{2\alpha} = +\infty$$

and this contradicts the fact that $a(\lambda_0) \in K^{\alpha,\beta}_{N \times N}$. Hence, the first equality in (4.14) is proved. The second equality in (4.14) can be proved by analogy. $\qquad\square$

4.4. **Proof of Theorem 1.2.**

Proof. The proof is developed similarly to the proofs of [14, Theorem 1.5], [15, Theorem 1.4], [16, Theorem 2.2] with some modifications. For the convenience of the reader, we provide some details.

Without loss of generality, we can suppose that $\max\{\alpha, \beta\} > 1/2$ (otherwise $\max\{\alpha, \beta\} \leq 1/2$ and $\alpha + \beta \geq 1$ imply that $\alpha = \beta = 1/2$ and this is exactly the case of Theorem 1.1).

Let Ω be a bounded open set containing the set $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$ on the closure of which f is analytic. From (2.2) and [6, Theorem 7.20] it follows that Ω contains the spectrum (eigenvalues) of $T_n(a)$ for all sufficiently large n. Further, Corollary 2.4 and Theorem [6, Theorem 2.94] imply that the spectrum of a in $K^{\alpha,\beta}_{N \times N}$ is contained in Ω. Hence $f(a) \in K^{\alpha,\beta}_{N \times N}$ and $f(T_n(a))$ is well defined whenever f is analytic on $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$.

One can choose a closed neighborhood Σ of its boundary $\partial\Omega$ such that f is analytic on Σ and $\Sigma \cap (\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})) = \emptyset$. If $\lambda \in \Sigma$, then $T(a) - \lambda I = T[a - \lambda]$ and $T(\widetilde{a}) - \lambda I = T[(a - \lambda)^\sim]$ are invertible on H^2_N. From Theorem 2.6 it follows that $(a - \lambda)^{-1} \in K^{\alpha,\beta}_{N \times N}$ and that $a - \lambda$ admits canonical right and left Wiener-Hopf factorizations

$$a - \lambda = u_-(\lambda)u_+(\lambda) = v_+(\lambda)v_-(\lambda)$$

in the algebra $K^{\alpha,\beta}_{N \times N}$. Since $a - \lambda : \Sigma \to K^{\alpha,\beta}_{N \times N}$ is a continuous function with respect to λ, from Lemma 2.5 and Theorem 2.2 we that these canonical factorizations can be chosen so that the functions

$$u^{\pm 1}_-, v^{\pm 1}_- : \Sigma \to (K^{\alpha,\beta} \cap \overline{H^\infty})_{N \times N}, \quad u^{\pm 1}_+, v^{\pm 1}_+ : \Sigma \to (K^{\alpha,\beta} \cap H^\infty)_{N \times N}$$

are continuous with respect to $\lambda \in \Sigma$.

Put $b(\lambda) := v_-(\lambda)u_+^{-1}(\lambda)$ and $c(\lambda) := u_-^{-1}(\lambda)v_+(\lambda)$. By Lemma 4.2,

$$(4.19) \qquad H[b(\lambda)]H[\widetilde{c(\lambda)}] \in \mathcal{C}_1(H_N^2)$$

for all $\lambda \in \Sigma$. On the other hand, since $(a-\lambda)^{-1} \in K_{N\times N}^{\alpha,\beta}$, we also have $(\widetilde{a}-\lambda)^{-1} = [(a-\lambda)^\frown]^{-1} \in K_{N\times N}^{\alpha,\beta}$. Then from Lemma 4.2 it follows that

$$(4.20) \qquad H[a-\lambda]H[(\widetilde{a}-\lambda)^{-1}] \in \mathcal{C}_1(H_N^2)$$

for all $\lambda \in \Sigma$. From (2.2), (4.19)–(4.20), and Theorem 3.2 we conclude that

$$(4.21) \qquad \begin{aligned} \det T_n(a-\lambda) &= G(a-\lambda)^{n+1} \det T[a-\lambda]T[(a-\lambda)^{-1}] \\ &\quad \times \det\big(I - Q_n H[b(\lambda)]H[\widetilde{c(\lambda)}]Q_n\big) \end{aligned}$$

for all $\lambda \in \Sigma$ and all $n \in \mathbb{N}$.

From Lemmas 4.3 and 4.4 it follows that there exists a number $n_0 \in \mathbb{N}$ such that

$$(4.22) \quad \big\|Q_n H[b(\lambda)]H[\widetilde{c(\lambda)}]Q_n\big\|_{\mathcal{C}_1(H_N^2)} \leq \frac{L\left(\sup\limits_{\lambda\in\Sigma} R_n^+\big(b(\lambda)\big)\right)\left(\sup\limits_{\lambda\in\Sigma} R_n^-\big(c(\lambda)\big)\right)}{n^{\alpha+\beta-1}} < 1$$

for all $\lambda \in \Sigma$ and all $n \geq n_0$. Here L is a positive constant depending only only on α, β an N. Combining (4.22) and Lemma 3.1, we arrive at the estimate

$$(4.23) \qquad \begin{aligned} &\big|\log\det\big(I - Q_n H[b(\lambda)]H[\widetilde{c(\lambda)}]Q_n\big)\big| \\ &\leq \frac{2L}{n^{\alpha+\beta-1}}\left(\sup\limits_{\lambda\in\Sigma} R_n^+\big(b(\lambda)\big)\right)\left(\sup\limits_{\lambda\in\Sigma} R_n^-\big(c(\lambda)\big)\right) \end{aligned}$$

for all $\lambda \in \Sigma$ and all $n \geq n_0$.

From (4.21), (4.23), and Lemma 4.4 we conclude that

$$(4.24) \qquad \begin{aligned} \log\det T_n(a-\lambda) &= (n+1)\log G(a-\lambda) \\ &\quad + \log\det T[a-\lambda]T[(a-\lambda)^{-1}] + o(n^{1-\alpha-\beta}) \end{aligned}$$

as $n \to \infty$ uniformly with respect to $\lambda \in \Sigma$. Hence we can differentiate both sides with respect to λ, multiply by $f(\lambda)$, and integrate over $\partial\Omega$. The rest of the proof is the repetition of Widom's arguments [26, Theorem 6.2] (see also [6, Section 10.90] and [5, Theorem 5.6]) with $o(1)$ replaced by $o(n^{1-\alpha-\beta})$. $\qquad\square$

Remark 4.5. Formula (4.24) for $\lambda = 0$ and $a \in K_{N\times N}^{\alpha,\alpha}$ with $\alpha > 1/2$ was obtained by Silbermann [23] by using methods of [3] (see also [4, Sections 6.15–6.23] and [6, Sections 10.34–10.37]). On the other hand, for $\alpha+\beta = 1$, formula (4.24) with $\lambda = 0$ was proved by Böttcher and Silbermann [4, Theorem 6.11] and [6, Theorem 10.30].

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